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Segments of Circles

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Segments of Circles:
The Arc, Chord, Radius, Height, Angle, Apothem, and Area

Suppose you have a segment of a circle, bounded by an arc of the circle and the chord subtending it. Let the length of the arc be **s**, the length of the chord be **c**, the radius of the circle be **r**, the distance from the midpoint of the chord to the midpoint of the arc be **h** (the height), the measure in radians of the central angle subtending the arc be **theta**, the distance from the midpoint of the chord to the center of the circle (the apothem) be **d**, and the area be **K**.

(Recall that 1 degree = $\pi/180$ radians.)

For a diagram, see the Ask Dr. Math FAQ on the segment of a circle.

No one of these values is enough to determine the other six. Any two of them, however, are sufficient to determine the remaining five.

If you are given two of these values, how do you calculate the others? There are 21 cases. Click on the data you know from the table below.

s, c	s, r	s, h	s, theta	s, d	s, K
	c, r	c, h	c, theta	c, d	c, K
		r, h	r, theta	r, d	r, K
			h, theta	h, d	h, K

theta, d theta, K
d, K

Reader Dan Erb has programmed these cases into an Excel Spreadsheet, which is available for download.

Case 1: You know **s** and **c**. Then solve

```
c/s = sin(x)/x,
```

for x, which must be done numerically (see below). Then

```
theta = 2x,
r     = s/theta,
d     = r cos(x),
h     = r - d,
K     = r²[theta-sin(theta)]/2.
```

Case 2: You know **s** and **r**. Then

```
theta = s/r,
c     = 2r sin(theta/2),
d     = r cos(theta/2),
h     = r - d,
K     = r²[theta-sin(theta)]/2.
```

Case 3: You know **s** and **h**. Then solve

```
2h/s = (1-cos[x])/x,
```

for x, which must be done numerically (see below). Then

```
theta = 2x,
r     = s/theta,
c     = 2r sin(x),
d     = r - h,
K     = r²[theta-sin(theta)]/2.
```

Case 4: You know **s** and **theta**. Then

```
r = s/theta,
c = 2r sin(theta/2),
d = r cos(theta/2),
h = r - d,
K = r²[theta-sin(theta)]/2.
```

Case 5: You know **s** and **d**. Then solve

```
2d/s = cos(x)/x,
```

for x, which must be done numerically (see below). Then

```
theta = 2x,
r     = s/theta,
c     = 2r sin(x),
h     = r - d,
K     = r²[theta-sin(theta)]/2.
```

Case 6: You know **s** and **K**. Then solve

```
2K/s² = (theta-sin[theta])/theta²
```

for theta, which must be done numerically (see below). Then

```
r = s/theta,
c = 2r sin(theta/2),
d = r cos(theta/2),
h = r - d.
```

Case 7: You know **c** and **r**. Then

```
theta = 2 arcsin(c/[2r]),
s     = r theta,
d     = r cos(theta/2),
h     = r - d,
K     = r²[theta-sin(theta)]/2.
```

Case 8: You know **c** and **h**. Then

```
r     = (c²+4h²)/(8h),
theta = 2 arcsin(c/[2r]),
s     = r theta,
d     = r - h,
K     = r²[theta-sin(theta)]/2.
```

Case 9: You know **c** and **theta**. Then

```
r = c/(2 sin[theta/2]),
s = r theta,
d = r cos(theta/2),
h = d - r,
K = r²[theta-sin(theta)]/2.
```

Case 10: You know **c** and **d**. Then

```
r     = sqrt(c²+4d²)/2,
h     = r - d,
theta = 2 arcsin(c/[2r]),
s     = r theta,
K     = r²[theta-sin(theta)]/2.
```

Case 11: You know **c** and **K**. Then solve

```
4K/c² = (theta-sin[theta])/(1-cos[theta])
```

for theta, which must be done numerically (see below). Then

```
r = c/(2 sin[theta/2]),
s = r theta,
d = r cos(theta/2)
h = r - d.
```

Case 12: You know **r** and **h**. Then

```
d     = r - h,
theta = 2 arccos(d/r),
c     = 2r sin(theta/2),
s     = r theta,
K     = r^2[theta-sin(theta)]/2.
```

Case 13: You know **r** and **theta**. Then

```
s = r theta,
d = r cos(theta/2),
h = r - d,
c = 2r sin(theta/2),
K = r^2[theta-sin(theta)]/2.
```

Case 14: You know **r** and **d**. Then

```
h     = r - d,
theta = 2 arccos(d/r),
c     = 2r sin(theta/2),
s     = r theta,
K     = r^2[theta-sin(theta)]/2.
```

Case 15: You know **r** and **K**. Then solve

```
2K/r^2 = theta - sin(theta)
```

for theta, which must be done numerically (see below). Then

```
s = r theta,
c = 2r sin(theta/2),
d = r cos(theta/2),
h = r - d.
```

Case 16: You know **h** and **theta**. Then

```
r = h/(1-cos[theta/2]),
d = r - h,
c = 2r sin(theta/2),
s = r theta,
```

```
K = r^2[theta-sin(theta)]/2.
```

Case 17: You know **h** and **d**. Then

```
r     = h + d,
theta = 2 arccos(d/r),
c     = 2r sin(theta/2)
s     = r theta,
K     = r^2[theta-sin(theta)]/2.
```

Case 18: You know **h** and **K**. Then solve

```
2K/h^2 = (theta-sin[theta])/(1-cos[theta/2])^2
```

for theta, which must be done numerically (see below). Then

```
r = sqrt(2K/[theta-sin(theta)]),
d = r - h,
c = 2r sin(theta/2)
s = r theta.
```

Case 19: You know **theta** and **d**. Then

```
r = d/(cos[theta/2]),
h = r - d,
c = 2d tan(theta/2),
s = r theta,
K = r^2[theta-sin(theta)]/2.
```

Case 20: You know **theta** and **K**. Then

```
r = sqrt(2K/[theta-sin(theta)]),
s = r theta,
d = r cos(theta/2),
h = r - d,
c = 2r sin(theta/2).
```

Case 21: You know **d** and **K**. Then solve

```
K/d² = (theta-sin[theta])/(1+cos[theta])
```

for theta, which must be done numerically (see below). Then

```
r = d/cos(theta/2),
h = r - d,
c = 2r sin(theta/2),
s = r theta.
```

Numerical Solutions

The equation f(x) = 0 can be solved numerically using (Sir Isaac) Newton's Method as follows.

Guess a starting value x(0). This will depend heavily on the form of f(x). Then for each n = 0, 1, 2, ..., compute

```
x(n+1) = x(n) - f[x(n)]/f'[x(n)],
```

where f'(x) is the derivative of f(x) with respect to x. Continue this until |x(n+1)-x(n)| is smaller than the accuracy sought. Then x(n+1) agrees with the actual root to at least that level of accuracy.

If the starting value for x(0) is close enough to a root, this will converge, and very rapidly. If x(0) is rather far from a root, this process may diverge, or exhibit other undesirable behavior.

To solve, for example, sin(x)/x = k for some constant k > 0 (Case 1 above), is the same as finding a root of the equation

```
f(x) = sin(x) - kx = 0.
```

This can be done using Newton's Method as follows.

Guess a starting value of

```
x(0) = sqrt(6-6k).
```

Then for each n = 0, 1, 2, ..., compute

```
x(n+1) = x(n) - (sin[x(n)]-kx(n))/(cos[x(n)]-k).
```

Example: Solve sin(x)/x = 3/4 to five decimal places of accuracy. We carry seven places of accuracy in our calculations:

n	x(n)	sin(x[n])	cos(x[n])	sin(x[n])/x(n)
0	1.2247449	0.9407193	0.3391860	0.7680941
1	1.2786882	0.9576389	0.2879717	0.7489229
2	1.2757074	0.9567763	0.2908250	0.7499967
3	1.2756981	0.9567736	0.2908338	0.7500000
4	1.2756981			

Since |x(4)-x(3)| < 0.000005, we have the answer correct to five decimal places, x = 1.27570.

There is also an infinite series which can solve sin(x)/x = k, 0 < x <= π. Let y = 1 - k. Then 0 < y < 1, and the following infinite series converges in this range:

```
x = sqrt(6y)(1 +
             3y/20 +
             321y^2/5600 +
             3197y^3/112000 +
             445617y^4/27596800 +
             1766784699y^5/179379200000 + ...).
```

This converges rapidly for large values of k = 1 - y.

Another series is

```
x = π/(1+k) -
    (π^3k^3/6)(1 -
             4k +
             [10+9π^2/20]k^2 -
             [20+16 π^2/5]k^3 +
             ...),

  = π/(1+k) -
```

```
(π³k³/[6(1+k)⁴]) -
(3π⁵k⁵/40)(1 - 64k/9 + ...).
```

This converges rapidly for small values of k.

Newton's Method works for (1-cos[x])/x = k (Case 3 above), with the initial start

```
x(0) = 2k,
```

and iteration

```
x(n+1) = x(n) - (cos[x(n)]+kx(n)-1)/(-sin[x(n)]+k).
```

Newton's Method works for cos(x)/x = k (Case 5 above) with the initial start

```
x(0) = sqrt(2+k²) - k,
```

and iteration

```
x(n+1) = x(n) - (cos[x(n)]-kx[n])/(-sin[x(n)]-k).
```

Newton's Method works for k = (x-sin[x])/x^2 (Case 6 above), with the initial start

```
x(0) = 6k,
```

and iteration

```
x(n+1) = x(n) - (kx(n)²-x(n)+sin[x(n)])/
                (2kx(n)-1+cos[x(n)]).
```

Newton's Method works for k = (t-sin[t])/(1-cos[t]) (Case 11 above), with the initial start

```
t(0) = 3k,
```

and iteration

```
t(n+1) = t(n) - (k cos[t(n)]-sin[t(n)]+t(n)-k)/
                  (1-k sin[t(n)]-cos[t(n)]).
```

Newton's Method works for k = x - sin(x) (Case 15 above), with the initial start

```
x(0) = (6k)^(1/3),
```

and iteration

```
x(n+1) = x(n) - [x(n)-sin(x(n))-k]/[1-cos(x(n))].
```

Newton's Method works for k = (t-sin[t])/(1-cos[t/2])^2 (Case 18 above), with the initial start

```
t(0) = 32/(3k),
```

and iteration

```
t(n+1) = t(n) - (k[1-cos(t(n)/2)]^2-t(n)+sin[t(n)])/
                  (k sin[t(n)/2]-k sin[t(n)]/2-1+cos[t(n)]).
```

Newton's Method works for k = (t-sin[t])/(1+cos[t]) (Case 21 above), with the initial start

```
t(0) = (12k)^(1/3),
```

and iteration

```
t(n+1) = t(n) - (t(n)-sin[t(n)]-k cos[t(n)]-k)/
                  (1-cos[t(n)]+k sin[t(n)]).
```

One Additional Situation

If you know the fraction of the area F = K/(πr^2), and you want to compute the fraction of the height h/(2r), this can also be done. That means you know 2Fπ = theta - sin(theta), and

you want to find theta. You must do this numerically. You can use Newton's Method for this, as well:

$$t[0] = (12F\pi)^{1/3},$$
$$t[n+1] = [\sin(t[n]) - t[n]\cos(t[n]) + 2F\pi] / [1 - \cos(t[n])],$$

for n = 0, 1, 2, 3, ... The sequence t[n] will converge to theta. Then

$$h/(2r) = [1 - \cos(\text{theta}/2)]/2.$$

- Robert L. Ward, for the Math Forum

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